Double-Line Harmony in a Sequent Setting

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Abstract: This paper concentrates on how to capture harmony in sequent calculi. It starts by considering a proposal made by Tennant and some objections to it which have been presented by Steinberger. Then it proposes a different analysis which makes use of a double-line presentation of sequent calculi in the style of Došen and it shows that this proposal is able to dismiss disharmonious operators without thereby adopting any global criterion.

Keywords: Proof-theoretic semantics, harmony, sequent calculi, double-line rules, inversion principles.

1 Introduction

Logical inferentialism maintains that the meaning of a logical operator \$ should be explained solely in terms of the rules of inference governing its behaviour. But, as shown by Prior's (1960) infamous operator $tonk$, not any set of rules introduces a meaningful operator. Thus, many different provisos have been provided to rule out tonkish operators, both in natural deduction (ND) and in sequent calculi (SC). One idea that has received much attention is the claim that the rules of inference for \$ must be in *harmony*, i.e. that there must be 'a certain consonance between the two aspects [i.e. introduction and elimination rules in ND and right and left rules in SC] of the use of a given form of expression' (Dummett, 1973, p. 397). There has been little agreement on how to make precise this intuitive notion of harmony. In the literature there have been four kinds of explication. We may talk of (i) *global harmony* when harmony is explicated in terms of conservativity and uniqueness (Belnap, 1962); (ii) *intrinsic harmony* when it is explicated in terms of reduction procedures (Dummett, 1991; Prawitz, 1965); (iii) *general elimination harmony* when the elimination rules can be

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read off the introduction rules (Negri & von Plato, 2015; Read, 2000); and (iv) *harmony as deductive equilibrium* when harmony is explicated directly in terms of an equilibrium between the two kinds of rules (Tennant, forth.).

One point of agreement between most proposals is that the rules governing an operator can be disharmonious in two different ways. The lack of consonance between the right and left rules in SC—or, equivalently, between the introduction and elimination rules in ND—may depend either (i) on the left rules being too strong with respect to the right ones, or (ii) on their being too weak. Following (Steinberger, 2011b) we talk of *S-disharmony* in the first case and of *W-disharmony* in the second one. A paradigmatic case of S-disharmony is Prior's *tonk* (▲), which is an operator having the left rules of conjunction and the right ones of disjunction.[2] A paradigmatic case of W-disharmony is its dual *knot* (▼), which is an operator having the left rule of disjunction and the right one of conjunction. A satisfactory analysis of harmony must rule out both *tonk* and *knot*.

Even though harmony is usually considered in the context of ND, when considering logics other than intuitionistic logic SC behave better. One advantage of SC is that they allow for multiple conclusions which are useful for some logics—e.g. for classical logic—and, possibly, essential for some others—e.g. for linear logic and for dual-intuitionistic logic. Another advantage is that SC allow to capture many modal logics which have no (known) formulation in ND. Furthermore, SC represent deductions more appropriately for those who take the concept of consequence as more fundamental than truth, (Schroeder-Heister, 2012). Thus, it is important to discuss the notion of harmony in SC.

In the context of SC, the analysis of harmony which is more widespread is the one in terms of global harmony. Belnap's seminal paper (1962) was precisely devoted to the introduction of global harmony in SC. Similar proposals have been given by Hacking (1979) for classical logic and, more recently, by Wansing (1998) for modal logics. Nevertheless, the global harmony of an operator \$ may depend on whether some other operator is present, and we prefer to analyse the harmony of \$ in terms of the rules governing its behaviour and independently of other operators—i.e. we opt for a so-called *local* analysis of harmony.

The local analyses in terms of intrinsic harmony and of general elimination harmony seems to be tailored for ND. Thus, this paper concentrates

[2]Usually *tonk* is defined by taking only rules $L\blacktriangle_2$ and $R\blacktriangle_1$ from Table 2. We prefer this equivalent and more symmetric version.

Table 1: $\{\vee, \exists\}$-fragment of LK

$$\frac{A_1, \Gamma \Longrightarrow \Delta \qquad A_2, \Gamma \Longrightarrow \Delta}{A_1 \vee A_2, \Gamma \Longrightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Longrightarrow \Delta, A_i}{\Gamma \Longrightarrow \Delta, A_1 \vee A_2}\, R\vee_i,\ i\in\{1,2\}$$

$$\frac{A[y/x], \Gamma \Longrightarrow \Delta}{\exists x A, \Gamma \Longrightarrow \Delta}\, L\exists,\ y \text{ fresh} \qquad \frac{\Gamma \Longrightarrow \Delta, A[t/x]}{\Gamma \Longrightarrow \Delta, \exists x A}\, R\exists$$

$$\frac{}{A \Longrightarrow A}\, \mathit{Ref} \qquad \frac{\Gamma \Longrightarrow \Delta, A \qquad A, \Gamma' \Longrightarrow \Delta'}{\Gamma, \Gamma' \Longrightarrow \Delta, \Delta'}\, \mathit{Cut} \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma', \Gamma \Longrightarrow \Delta, \Delta'}\, W$$

Table 2: Rules for ▲ and ▼

$$\frac{A_i, \Gamma \Longrightarrow \Delta}{A_1 \blacktriangle A_2, \Gamma \Longrightarrow \Delta}\, L\blacktriangle_i,\ i\in\{1,2\} \qquad \frac{\Gamma \Longrightarrow \Delta, A_i}{\Gamma \Longrightarrow \Delta, A_1 \blacktriangle A_2}\, R\blacktriangle_i,\ i\in\{1,2\}$$

$$\frac{A_1, \Gamma \Longrightarrow \Delta \qquad A_2, \Gamma \Longrightarrow \Delta}{A_1 \blacktriangledown A_2, \Gamma \Longrightarrow \Delta}\, L\blacktriangledown \qquad \frac{\Gamma \Longrightarrow \Delta, A_1 \qquad \Gamma \Longrightarrow \Delta, A_2}{\Gamma \Longrightarrow \Delta, A_1 \blacktriangledown A_2}\, R\blacktriangledown$$

on how to capture harmony as (local) deductive equilibrium in SC (as will be presented in Definition 1). It starts by considering a proposal made by Tennant (2010, forth.) and some objections to it which have been presented in (Steinberger, 2009, 2011a). Then it proposes a different analysis which makes use of a double-line presentation of SC in the style of (Došen, 1989) and it shows that this proposal, which is purely local, avoids Steinberger's objections to Tennant's proposal. Without loss of generality, we will consider only the $\{\vee, \exists\}$-fragment of Gentzen's LK with context-as-sets and we consider a standard first-order language, see (Negri & von Plato, 2001). Table 1 gives the rules of this calculus and Table 2 gives the rules of the operators *tonk* and *knot* which will be used here as test for disharmony.[3]

[3] All we will say works equally good for the single conclusion intuitionistic calculus LJ. The addition of negation to LK is not a problem to our approach.

2 Harmony in sequent calculus

Tennant analyses harmony in SC as a deductive equilibrium between left and right introduction rules for an operator.[4] Harmony is 'a kind of Nash equilibrium between introduction and elimination rules' (Tennant, forth., p. 22). Roughly, for a monadic \$, the idea is that $\$A$ has to be the strongest formula derivable by $R\$$ and the weakest one derivable by $L\$$ (and vice versa), or equivalently:

Definition 1 (H-DE) *If we take $L\$$ as primitive, then $R\$$ has to allow us to derive* no more *(*no less*) sequents than those already derivable from a possible premiss of $R\$$ (conclusion of $L\$$).*

Notice that H-DE contains a *no-gain* condition, which is given by its 'no more…' clause and which is meant to rule out S-disharmonious operators, and it contains a *no-loss* condition, which is given by its 'no less…' clause and which is meant to rule out W-disharmonious operators. This will be extremely important in section 3 of this paper.

Tennant (2010) makes this precise by requiring that the rules for \$ satisfy three constraints: *h̲armony*, *maximality* and *admissibility*. The constraint of h̲armony is based on the following Fregean definition of (logical) strength:

> a proposition ϕ is at least as strong as ψ iff the derivability of the sequent $\psi, \Gamma \Longrightarrow \Delta$ entails the derivability of $\phi, \Gamma \Longrightarrow \Delta$.

This allows Tennant to define h̲armony of \$, $h(R\$, L\$)$, as follows:

(S) If ψ satisfy the conditions on $\$A$ in rule $R\$$, then by making full use of $L\$$ and no use of $R\$$ we can show that $\$A$ is at least as strong as ψ.

(W) If ψ satisfy the conditions on $\$A$ in rule $L\$$, then by making full use of $R\$$ and no use of $L\$$ we can prove that $\$A$ is at least as weak as ψ.

It is immediate to notice that h̲armony holds for $\vee$ and $\exists$. To illustrate, if we take a ψ with the following right introduction rule: $\dfrac{\Gamma \Longrightarrow \Delta, A[t/x]}{\Gamma \Longrightarrow \Delta, \psi}\, {\scriptstyle R\psi}$, then we can prove that the rules for $\exists$ comply with (S) as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A[y/x] \Longrightarrow A[y/x]}\,{\scriptstyle Ref}}{A[y/x] \Longrightarrow \psi}\,{\scriptstyle R\psi} \qquad \psi, \Gamma \Longrightarrow \Delta}{A[y/x], \Gamma \Longrightarrow \Delta}\,{\scriptstyle Cut}}{\exists x A, \Gamma \Longrightarrow \Delta}\,{\scriptstyle L\exists} \tag{1}$$

[4] Tennant's (forth.) analysis is in terms of ND and he introduces a SC-version in his (2010) to show that Steinberger's (2009) objection goes wrong.

and, by taking a ψ with the following left rule: $\dfrac{A[y/x], \Gamma \Longrightarrow \Delta}{\psi, \Gamma \Longrightarrow \Delta}\,{\scriptstyle L\psi,\ y\ \text{fresh}}$, we can prove (W) as follows:

$$\dfrac{\dfrac{\dfrac{}{A[y/x] \Longrightarrow A[y/x]}\,{\scriptstyle \mathit{Ref}}}{A[y/x] \Longrightarrow \exists x A}\,{\scriptstyle R\exists} \qquad \exists x A, \Gamma \Longrightarrow \Delta}{\dfrac{A[y/x], \Gamma \Longrightarrow \Delta}{\psi, \Gamma \Longrightarrow \Delta}\,{\scriptstyle L\psi}}\,{\scriptstyle \mathit{Cut}} \tag{2}$$

The notion of $\underline{h}$armony eliminates some S-disharmonious operators, as it is witnessed by the fact that in trying to prove that (S) and (W) hold for *tonk* we cannot make full use of the rules for ▲—e.g. in a proof of (S) we can use only one of the rules $L▲_1$ and $L▲_2$:

$$\dfrac{\dfrac{\dfrac{}{A_i \Longrightarrow A_i}\,{\scriptstyle \mathit{Ref}}}{A_i \Longrightarrow \psi}\,{\scriptstyle R\psi_i} \qquad \psi, \Gamma \Longrightarrow \Delta}{\dfrac{A_i, \Gamma \Longrightarrow \Delta}{A_1 ▲ A_2, \Gamma \Longrightarrow \Delta}\,{\scriptstyle L▲_i}}\,{\scriptstyle \mathit{Cut}}$$

Moreover, $\underline{h}$armony eliminates some W-disharmonious operators, as it is witnessed by the fact that there is no way of proving either (S) or (W) for *knot*— e.g. if ψ satisfies the conditions on $A▼B$ in rule $L▼$, we have the following failed proof-attempt of (W)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Longrightarrow A}\,{\scriptstyle \mathit{ref}}}{A, B \Longrightarrow A}\,{\scriptstyle W} \qquad \dfrac{\dfrac{}{B \Longrightarrow B}\,{\scriptstyle \mathit{ref}}}{A, B \Longrightarrow B}\,{\scriptstyle W}}{A, B \Longrightarrow A▼B}\,{\scriptstyle R▼} \qquad A▼B, \Gamma \Longrightarrow \Delta}{\dfrac{A, B, \Gamma \Longrightarrow \Delta}{?}}\,{\scriptstyle \mathit{Cut}}$$

Thus, $\underline{h}$armony helps in determining harmonious operators, which is much in line with Tennant's aim. Nevertheless, he argues that this constraint is necessary but not sufficient for H-DE because it doesn't determine uniquely the rules of the operators. To wit, $\underline{h}$armony is satisfied not only by the rules for ordinary disjunction ($\vee$) but also by the ones for quantum disjunction ($\curlyvee$):

$$\dfrac{A_1 \Longrightarrow \Delta \qquad A_2 \Longrightarrow \Delta}{A_1 \curlyvee A_2 \Longrightarrow \Delta}\,{\scriptstyle L\curlyvee} \qquad \dfrac{\Gamma \Longrightarrow \Delta, A_i}{\Gamma \Longrightarrow \Delta, A_1 \curlyvee A_2}\,{\scriptstyle R\curlyvee_i} \tag{3}$$

In fact a proof that $\vee$ satisfies (S) and (W) is, at the same time, a proof that $\curlyvee$ satisfies them. Given that it is well known that it is problematic to have a calculus with both standard and quantum disjunction, Tennant wants to rule out one of the two operators, and, therefore, he ensures the unique determination of the operators of LK by requiring *maximality*:

> $R\$$ has to be the strongest right rule that is in $\underline{\text{h}}$armony with $L\$$;
> and $L\$$ the strongest left rule that is in $\underline{\text{h}}$armony with $R\$$.

This condition forces the adoption of standard disjunction and the dismissal of quantum disjunction since (i) these two operators have the same right rules and (ii) the rule $L\vee$ is stronger than $L\curlyvee$ in that any sequent derivable with the latter is derivable with the former, but not vice versa.

The joint requirement of $\underline{\text{h}}$armony and maximality eliminates all forms of W-disharmony. But, as shown in (Steinberger, 2009), they introduce some form of S-disharmony for the quantifiers because, instead of the existential quantifier $\exists$, they pick up the rogue quantifier $\mathit{3}$, whose rules are

$$\frac{A[t/x],\Gamma \Longrightarrow \Delta}{\mathit{3}xA,\Gamma \Longrightarrow \Delta}\,{\scriptstyle L\mathit{3}} \qquad \frac{\Gamma \Longrightarrow \Delta, A[t/x]}{\Gamma \Longrightarrow \Delta, \mathit{3}xA}\,{\scriptstyle R\mathit{3}} \tag{4}$$

with no variable condition on the left rule. Given that the variable restriction on $L\exists$ played no major role in the proofs given in (1) and (2), these proofs show also that $\mathit{3}$ satisfies (S) and (W). Moreover, maximality forces us to adopt $\mathit{3}$ in place of $\exists$ given that (i) they have the same right rule and (ii) $\mathit{3}$ has a stronger left rule which allows to derive the schematic sequent $A[s/x] \Longrightarrow A[t/x]$ for any pair of terms s,t. This last fact means that $\mathit{3}$ is S-disharmonious in that it has a left rule that is unduly too permissive with respect to its right rule.

Tennant (2010) argues that the S-disharmonious operator $\mathit{3}$ is ruled out by the *admissibility* constraint: the existence of a syntactic proof of cut-elimination.[5] This constraint rules out $\mathit{3}$ since the rules of this operator do not satisfy even the weaker requisite of *cut-inductiveness*—i.e. eliminability of a cut with cut formula principal in both premisses—as it is witnessed by the fact that the following cut

$$\dfrac{\dfrac{\Gamma \Longrightarrow \Delta, A[s/x]}{\Gamma \Longrightarrow \Delta, \mathit{3}xA}\,{\scriptstyle R\mathit{3}} \qquad \dfrac{A[t/x],\Pi \Longrightarrow \Sigma}{\mathit{3}xA,\Pi \Longrightarrow \Sigma}\,{\scriptstyle L\mathit{3}}}{\Gamma,\Pi \Longrightarrow \Delta,\Sigma}\,{\scriptstyle Cut} \tag{5}$$

[5] We have transformed Tennant's request for a proof of cut-admissibility into the request for a proof of cut-elimination. The two requisites are equivalent in the present setting and we have chosen to follow Gentzen in taking cut as a primitive rule of inference of LK.

is not eliminable whenever $s \neq t$ and t occurs free in Π, Σ.

Steinberger (2011a) argues that the problems with admissibility are that (i) it makes the notion of $\underline{\text{h}}$armony redundant and that (ii) it is a global criterion. It is known that cut-inductiveness disposes of S-disharmonious operators—such as $\blacktriangle$—and maximality disposes of W-disharmonious ones—such as $\blacktriangledown$. Thus, we can replace the two constraints of $\underline{\text{h}}$armony and admissibility with the local one of cut-inductiveness. Steinberger claims that an analysis of H-DE in terms of cut-inductiveness and maximality behaves better than Tennant's one in that (i) it eliminates the same disharmonious operators, but (ii) it is not redundant and (iii) it is does not contain the global requisite of admissibility.

Even though we may agree that cut-inductiveness behaves better than $\underline{\text{h}}$armony+admissiblity, we do not think that Steinberger's analysis is purely local because maximality is not a local criterion. If we consider again the case of $\vee$ and $\curlyvee$, we see immediately that maximality rules out $\curlyvee$ precisely because of the existence of the stronger $\vee$—i.e. it goes global in that it sanctions an operator as disharmonious because of the existence of another one. Dicher (2016) argues that no local (extensional) criterion is able to rule out $\curlyvee$. If this is true, we have to look for an analysis which rules out both S- and W-disharmonious operators without thereby ruling out also $\curlyvee$.

To sum up, we are looking for a purely local analysis of harmony as H-DE in SC. Tennant (2010) analyses H-DE in terms of $\underline{\text{h}}$armony, maximality and admissibility, and Steinberger (2011a) in terms of maximality and cut-inductiveness. H-DE has the following advantages w.r.t. other well known explications of harmony: (i) in banning both rules that are too strong and rules that are too weak it allows to determine one rule from the other in a unique way—this doesn't hold for intrinsic harmony; and (ii) it is in principle a local analysis since it refers only to the rules of the operator under consideration—this doesn't hold for global harmony.

We believe that H-DE is on the right track. However, we also believe that a purely local analysis of harmony as deductive equilibrium has to differ from Tennant's and from Steinberger's proposals.

3 Double-Line harmony

We want to capture H-DE in a way that (i) ensures the unique determination of one rule from the other without thereby (ii) having to adopt the global requirement of maximality. In ND this is feasible thanks to the *generalized* *inversion principle* (GIP): 'whatever follows from the direct grounds for deriving a proposition must follow from that proposition' (Negri & von Plato, 2001, p. 6). Already in (Prawitz, 1965) the inversion principle has been the essential element for defining harmony in ND. Even though Prawitz's formulation is not strong enough to ensure the unique determination of the elimination rules, GIP allows us to determine the (generalized) elimination rules as unique functions of the introduction rules (Negri & von Plato, 2015, p. 243); see also (Moriconi & Tesconi, 2008) on inversion principles. This fact lies at the core of general elimination harmony. We are now going to show how to give an explication of H-DE in SC which exploits something like GIP.

Došen (1989) introduces double-line rules to show the logicality of operators, where a double-line rule $\frac{\mathcal{S}_1}{\overline{\mathcal{S}_2}}\,{\uparrow\downarrow\mathcal{R}}$ is a rule that can be applied both in downward ($\downarrow \mathcal{R}$) and in upward ($\uparrow \mathcal{R}$) direction. As emphasised in (Došen, 2015, p. 151), the possibility of formulating SC via double-line rules is intimately related with the inversion principle. Thus, it should be possible to express H-DE by means of double-line rules. Nevertheless, as shown in (Bonnay & Simmenauer, 2005), formulating SC in terms of double-line rules does not in itself capture any notion of harmony. Roughly, the problem is that, unless we uses $G3$-style calculi, each operator has only either the left or the right rule that is invertible. The source of this problem points directly to its solution: we have to exploit the possibility of formulating either $L\$$ or R$ as a double-line rule as a kind of GIP that allows to determine the unique set of rules $R\$$ or L$ that are in deductive equilibrium with it. This idea can be captured formally as follows:

Definition 2 (dl-harmony) *The rules for an operator* $\$$ *are in* dl-harmony *whenever either* $L\$$ *or* $R\$$ *is the unique set of rules that has the same deductive strength of the* $\uparrow$ *direction of the (respectively right or left) double-line rule for* $\$$. *More precisely*

(dl-S) *we have to derive an arbitrary instance of (some)* $L\$$ *(resp.* $R\$$*) from any instance of* $\$ \uparrow$*; and*

(dl-W) *we have to derive an arbitrary instance of* $\$ \uparrow$ *from an instance of (some)* $L\$$ *(resp.* $R\$$*).*

Notice that dl-harmony contains both a no-gain condition—given by dl-S—and a no-loss condition—given by dl-W—and that these two conditions are precisely the constituents of H-DE. Thus, we propose that the rules for $ are in H-DE whenever they satisfy dl-harmony.

In order to substantiate our claim, we will show (i) that the operators of the $\{\vee, \exists\}$-fragment of LK satisfy dl-harmony, and (ii) that none of the disharmonious operators ▲, ▼, and $\mathit{3}$ satisfies it. In order to prove (i), let us consider the following double-line rules for $\vee$ and for $\exists$:

$$\frac{A_1, \Gamma \Longrightarrow \Delta \qquad A_2, \Gamma \Longrightarrow \Delta}{\overline{A_1 \vee A_2, \Gamma \Longrightarrow \Delta}}\ {\scriptstyle \uparrow\downarrow\vee} \qquad \frac{A[y/x]\Gamma \Longrightarrow \Delta}{\overline{\exists x A, \Gamma \Longrightarrow \Delta}}\ {\scriptstyle \uparrow\downarrow\exists},\ y \text{ fresh} \tag{6}$$

For $\vee$ we have to prove that the bottom-up rule $\uparrow \vee$ is interderivable with the rules $R\vee_i$.[6] We have the following proof of the no-gain condition dl-S:

$$\frac{\Gamma \Longrightarrow \Delta, A_1 \qquad \dfrac{\dfrac{}{A_1 \vee A_2 \Longrightarrow A_1 \vee A_2}\ {\scriptstyle ref}}{A_1 \Longrightarrow A_1 \vee A_2}\ {\scriptstyle \uparrow\vee}}{\Gamma \Longrightarrow \Delta, A_1 \vee A_2}\ {\scriptstyle Cut} \tag{7}$$

and we have the following proof of the no-loss condition dl-W:

$$\frac{\dfrac{\dfrac{}{A_1 \Longrightarrow A_1}\ {\scriptstyle ref}}{A_1 \Longrightarrow A_1 \vee A_2}\ {\scriptstyle R\vee_1} \qquad A_1 \vee A_2, \Gamma \Longrightarrow \Delta}{A_1, \Gamma \Longrightarrow \Delta}\ {\scriptstyle Cut} \tag{8}$$

When we come to the proof of the no-loss condition dl-S for the existential quantifier, we have to make essential use of the facts that (i) the double-line rule has a variable condition and that (ii) the rule of substitution of terms for variables[7] is (height-preserving) admissible in LK. The proof goes as follows:

$$\frac{\Gamma \Longrightarrow \Delta, A[t/x] \qquad \dfrac{\dfrac{\dfrac{}{\exists x A \Longrightarrow \exists x A}\ {\scriptstyle Ref}}{A[y/x] \Longrightarrow \exists x A}\ {\scriptstyle \uparrow\exists}}{A[t/x] \Longrightarrow \exists x A}\ {\scriptstyle [t/y]}}{\Gamma \Longrightarrow \Delta, \exists x A}\ {\scriptstyle Cut} \tag{9}$$

Finally, we have the following proof of the no-loss condition dl-W for $\exists$:

$$\frac{\dfrac{\dfrac{}{A[y/x] \Longrightarrow A[y/x]}\ {\scriptstyle Ref}}{A[y/x] \Longrightarrow \exists x A}\ {\scriptstyle R\exists} \qquad \exists x A, \Gamma \Longrightarrow \Delta}{A[y/x], \Gamma \Longrightarrow \Delta}\ {\scriptstyle Cut} \tag{10}$$

In order to prove (ii), we consider the disharmonious operators ▲, ▼, and $\mathit{3}$. For the S-disharmonious ▲, it is immediate to notice that it does

[6] Without loss of generality, we consider only the 'left half' of $\uparrow \vee$ and $R\vee_1$.

[7] With the usual proviso to avoid capture of free variables.

not satisfy dl-harmony because there is no single double-line rule that determines it. Suppose that we take the double-line version of either $R\blacktriangle_1$ or of $R\blacktriangle_2$, that is we take either

$$\frac{\Gamma \Longrightarrow \Delta, A_1}{\Gamma \Longrightarrow \Delta, A_1 \blacktriangle_1 A_2}\ {\scriptstyle \uparrow\downarrow\blacktriangle_1} \qquad \text{or} \qquad \frac{\Gamma \Longrightarrow \Delta, A_2}{\Gamma \Longrightarrow \Delta, A_1 \blacktriangle_2 A_2}\ {\scriptstyle \uparrow\downarrow\blacktriangle_2} \tag{11}$$

then with either rule, even if we can prove the no-loss condition, we cannot prove the no-gain condition. If we start with $\uparrow\downarrow\ \blacktriangle_1$ we can prove the no-gain condition only for $L\blacktriangle_1$ and, vice versa, if we start with $\uparrow\downarrow\ \blacktriangle_2$ we can prove the no-gain condition only for $L\blacktriangle_2$. This shows that the only left rules that are in dl-harmony—i.e. interderivable—with the rules in (11) are, respectively

$$\frac{A_1, \Gamma \Longrightarrow \Delta}{A_1 \blacktriangle_1 A_2, \Gamma \Longrightarrow \Delta}\ {\scriptstyle L\blacktriangle_1} \qquad \text{and} \qquad \frac{A_2, \Gamma \Longrightarrow \Delta}{A_1 \blacktriangle_2 A_2, \Gamma \Longrightarrow \Delta}\ {\scriptstyle L\blacktriangle_2}$$

and in neither case we determine the disharmonious operator $\blacktriangle$. These sets of rules determine, respectively, the harmonious first-projection operator $\blacktriangle_1$ and second-projection operator $\blacktriangle_2$. Roughly, the paradoxicality of *tonk* is explained as an equivocation between two distinct projection operators.

For the W-disharmonious operator $\blacktriangledown$, the following problem would arise with the double-line version of any of its rules. Suppose that we start with the double-line version of its right rule

$$\frac{\Gamma \Longrightarrow \Delta, A_1 \qquad \Gamma \Longrightarrow \Delta, A_2}{\Gamma \Longrightarrow \Delta, A_1 \blacktriangledown A_2}\ {\scriptstyle \uparrow\downarrow\blacktriangledown} \tag{12}$$

then we can prove that $L\blacktriangledown$ satisfies the no-gain condition but not that it satisfy the no-loss condition since we are at best able to prove

$$\dfrac{\Gamma \Longrightarrow \Delta, A_1 \blacktriangledown A_2 \qquad \dfrac{\dfrac{\dfrac{}{A_1 \Longrightarrow A_1}\,{\scriptstyle Ref}}{A_1 \Longrightarrow A_1, A_2}\,{\scriptstyle W} \qquad \dfrac{\dfrac{}{A_2 \Longrightarrow A_2}\,{\scriptstyle Ref}}{A_2 \Longrightarrow A_1, A_2}\,{\scriptstyle W}}{A_1 \blacktriangle A_2 \Longrightarrow A_1, A_2}\,{\scriptstyle L\blacktriangledown}}{\Gamma \Longrightarrow \Delta, A_1, A_2}\,{\scriptstyle Cut}$$

whose conclusion is not, and cannot be transformed into, the desired conclusion of $\blacktriangledown$. We conclude that also $\blacktriangledown$ is sanctioned as disharmonious by dl-harmony.[8]

[8] Without using the rule of weakening, the rule $\uparrow\downarrow\ \blacktriangledown$ given in (12) would determine the left rules of conjunction of LK, thus transforming $\blacktriangledown$ into (a notational variant of) conjunction.

Finally, we consider the S-disharmonious rogue quantifier $\mathit{3}$. The double-line rule $\uparrow\downarrow \mathit{3}$ is like the one given in (6) for the existential quantifier except that it does not have the variable restriction.[9] On the one hand, this operator satisfies the no-loss condition of dl-harmony. In fact the proof of dl-W given in (10) for $\exists$ made no use of the variable condition and, therefore, it works also for $\mathit{3}$. On the other hand, $\mathit{3}$ does not satisfy the no-gain condition and, therefore, is not in dl-harmony. This can be shown as follows. First, we reduce the search-space by noticing that if dl-S holds for $\mathit{3}$, it has to be provable as we did in (9) for $\exists$. That is, we have to transform

$$\dfrac{\Gamma \Longrightarrow \Delta, A[t/x] \qquad \dfrac{\dfrac{\dfrac{}{\mathit{3}xA \Longrightarrow \mathit{3}xA}\,\scriptstyle{Ref}}{A[y/x] \Longrightarrow \mathit{3}xA}\,\uparrow \mathit{3}}{(A[y/x])[t/y] \Longrightarrow (\mathit{3}xA)[t/y]}\,\scriptstyle{[t/y]}}{\Gamma \Longrightarrow \Delta, (\mathit{3}xA)[t/y]}\,\scriptstyle{Cut}$$

into a proof showing that $\mathit{3}$ satisfies dl-S. But this would be possible only if either y is t or if y does not occur free in $\mathit{3}xA$.[10] The latter assumption is feasible (without loss of generality) just in case $\mathit{3}$ satisfies the same variable condition as $\exists$. The former assumption—i.e. the idea of exemplifying $\uparrow \mathit{3}$ directly with the term t and not with an arbitrary term y—does not work because in proving the no-gain condition we cannot rely on a specific instance of $\uparrow \mathit{3}$. It might be objected that the instance of $\uparrow \mathit{3}$ which concludes $A[t/x] \Longrightarrow \mathit{3}xA$ is a legal one. Nevertheless, we do not accept it because the double-line rule for $\mathit{3}$ talks of all terms and not of a particular one and, therefore, in proving dl-harmony we have to apply it in all its generality. In a sense the quantifiers without variable condition that can be shown to be in dl-harmony are, for any given t, the innocuous t-specific quantifier $\mathit{3}^t$ which is like $\mathit{3}$ save that its rules are applicable only for the given term t. These rules give us an harmonious and meaningful operator, whose meaning implies that $A[t/x]$ is equivalent to $\mathit{3}^t xA$.

We have thus shown that dl-harmony gives the expected results—i.e. it is satisfied by $\vee$ and $\exists$ and it is not satisfied by ▲, ▼, and $\mathit{3}$. One nice aspect of dl-harmony is that it has a no-gain condition—i.e. dl-S—that rules out S-disharmonious operators such as ▲ and $\mathit{3}$ and it has a no-loss condition—i.e. dl-W—that rules out W-disharmonious operators such as ▼. Moreover

[9]Nothing essential relies on taking the double-line version of the left rule; by taking the right one we obtain the rogue universal quantifier.

[10]Notice that these two are the very same moves that would transform (5) into an effective proof of cut-inductiveness for $\mathit{3}$. Notice also that the admissibility of the rule of substitution depends essentially on the variable restriction of the rules for the quantifiers, see (Negri & von Plato, 2001, p. 69).

dl-harmony is a purely local requisite in that it relies only on the rules governing the operator under examination and on the structural rules of Cut and of Reflexivity. Notice that these two structural rules are necessary to prove its no-gain condition. For the no-loss condition it would be enough to have these rules as admissible. For example we could have proved dl-W also in the calculus LK without the rule of Cut since this rule is admissible in it. On the other hand, these two structural rules are not eliminable from the proofs of dl-S. These considerations show that dl-harmony could not hold for the operators of non-reflexive or non-transitive substructural logics. We don't take this as a severe limitation.

4 Conclusions

It has been shown that it is possible to give a purely local analysis of harmony as H-DE in SC by imposing the requisite of dl-harmony. As opposed to both Tennant's and Steinberger's proposals, the requisite of dl-harmony guarantees that the quantum disjunction operator $\curlyvee$ is harmonious. This happens because the proofs given in (7) and (8) still work if we replace the rule $\uparrow\downarrow\vee$ by the following one

$$\frac{A_1 \Longrightarrow \Delta \qquad A_2 \Longrightarrow \Delta}{A_1 \curlyvee A_2 \Longrightarrow \Delta}\ {\scriptstyle \uparrow\downarrow\curlyvee}$$

where the left-context is empty. This will be taken as a limitation of our approach by anyone who is inclined to consider $\curlyvee$ as (intuitively) disharmonious. We are not disturbed by this since we take $\curlyvee$ as a perfectly harmonious operator. Its only problem might be that it interacts badly with $\vee$, but for anyone who takes harmony to be a purely local matter, as we did, this should not be relevant for assessing whether it is harmonious or not.[11]

All in all, the analysis of harmony as H-DE in terms of dl-harmony differs from both Tennant's and Steinberger's in that it is a purely local analysis which is based on the inversion principle. Neither the requisite of maximality nor that of admissibility are acceptable in a purely local analysis of harmony. Notice that even if Tennant and Steinberger were to agree that $\curlyvee$ is harmonious, they cannot drop the global requisites from their analyses since (i) Tennant needs admissibility to rule out *3* and (ii) Steinberger needs maximality to rule out ▼. This is not intended by us as a criticism of their

[11] See Dicher (2016) for a critical presentation of many proposals that disagree with us on this point.

proposals inasmuch as they have not asked for a purely local analysis and, therefore, can use global requisites.

Like other approaches based on GIP, the present proposal satisfies a notion of unique determination which differs from the one given by maximality. With maximality unique determination holds in the sense that there cannot be two harmonious operators sharing some rule, as it happens for $\vee$ and $\curlyvee$. With dl-harmony unique determination holds in the sense that starting from the double-line rule governing an operator we determine one unique set of rules that are in harmony with it. A more comprehensive analysis of the relationship between dl-harmony and other concepts of harmony which are based on GIP—such as the ND-based ones in terms of general elimination harmony (Negri & von Plato, 2015; Read, 2000) and the SC-based ones in terms of so-called reflection principles (Sambin, Battilotti, & Faggian, 2000; Schroeder-Heister, 2007)—goes beyond the scope of this paper and we leave it to future research.

The present approach to harmony can be extended beyond the $\{\vee, \exists\}$-fragment of LK. At the present stage it is already clear that it can be extended to logics containing any additive or multiplicative operator of linear logic and to their extensions with the structural rules of weakening and contraction. It is also clear that it cannot be extended to Tennant's core logic ($\mathbb{C}$); because, from the present perspective, the operators of $\mathbb{C}$ are not harmonious in that they are not determinable according to GIP. For example, the rules for conjunction in $\mathbb{C}$ are the right rule for multiplicative conjunction and (a version of) the left rules for additive conjunction, neither of which is invertible. One natural future line of research is to give dl-harmonious rules for the operators of many modal logics by means of generalizations of Gentzen's SC such as display logics (Wansing, 1998) and labelled calculi (Negri & von Plato, 2015).

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Double-Line Harmony in a Sequent Setting

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